

A Remark on Oka's Coherence without Weierstrass' Preparation Theorem and the Oka Theory*

Junjiro Noguchi[†]

The University of Tokyo, Emeritus

Abstract

The proofs of Oka's Coherence Theorems are based on Weierstrass' Preparation (division) Theorem. Here we observe that a Weak Coherence of Oka proved without Weierstrass' Preparation (division) Theorem, but only with *power series expansions* is sufficient to prove Oka's Jôku-Ikô and hence Cousin I, II, holomorphic extensions, and Levi's Problem, as far as the domain spaces are non-singular. The proof of the Weak Coherence of Oka is almost of linear algebra. We will present some new or simplified arguments in the proofs.

1 Introduction and weak coherence of Oka

K. Oka [22], [23] proved three fundamental coherence theorems for

First: the sheaf $\mathcal{O} := \mathcal{O}_{\mathbf{C}^n}$ of germs of holomorphic functions on \mathbf{C}^n ,

Second: the geometric ideal sheaf $\mathcal{I}\langle A \rangle$ of an analytic subset A ,

Third: the normalization of the structure sheaf of a complex space,

where for the second, H. Cartan [3] gave his own proof based on Oka [22] (cf. [12] Chap. 9). The proofs of those coherence theorems rely on Weierstrass' Preparation (division) Theorem.

The purpose of this paper is to remark that a weak coherence theorem (Theorem 1.2 below) proved *not with Weierstrass' Preparation Theorem*, but *only with power series expansions* suffices to solve Cousin I, II Problems, $\bar{\partial}$ -equation (for functions), holomorphic extensions, and Levi's Problem (see Theorem 4.9 and §4.3).

*AMC2010: 32A99; 32E30

[†]Research supported in part by Grant-in-Aid for Scientific Research (C) 15K04917.

Let Ω denote a domain of \mathbf{C}^n with the structure sheaf $\mathcal{O} = \mathcal{O}_\Omega$. For a holomorphic function $f \in \mathcal{O}(\Omega)$ we write $\underline{f} \in \Gamma(\Omega, \mathcal{O})$ for the induced sheaf-section of \mathcal{O} and \underline{f}_z for the germ of f at $z \in \Omega$. Let \mathcal{F} be an analytic sheaf on Ω , and let $\xi_j \in \Gamma(\Omega, \mathcal{F})$, $1 \leq j \leq q$, be finitely many sections on Ω . Then the relation sheaf $\mathcal{R}(\xi_1, \dots, \xi_q)$ of $\{\xi_j\}_{j=1}^q$ is a subsheaf of \mathcal{O}^q consisting of those germ-vectors $(\underline{f}_1, \dots, \underline{f}_q) \in \mathcal{O}_z^q$ such that

$$(1.1) \quad \underline{f}_1 \xi_1(z) + \dots + \underline{f}_q \xi_q(z) = 0, \quad z \in \Omega.$$

Now we formulate:

Theorem 1.2 (Weak Coherence of Oka). *Let $S \subset \Omega$ be a complex submanifold.¹⁾*

- (i) *The geometric ideal sheaf $\mathcal{I}\langle S \rangle$ is locally finite.*
- (ii) *Let $\{\underline{\sigma}_j \in \Gamma(\Omega, \mathcal{I}\langle S \rangle) : 1 \leq j \leq N\}$ be a finite generator system of $\mathcal{I}\langle S \rangle$ on Ω with $\underline{\sigma}_j \in \mathcal{O}(\Omega)$: i.e.,*

$$\mathcal{I}\langle S \rangle = \sum_{j=1}^N \mathcal{O} \cdot \underline{\sigma}_j.$$

Then, the relation sheaf $\mathcal{R}(\underline{\sigma}_1, \dots, \underline{\sigma}_N)$ is locally finite.

We give a proof of this theorem in §2. In §3 we will apply it to prove Oka's Jôku-Ikô, and then we will give a unified proof for Cousin I, II Problems, and $\bar{\partial}$ -equation for functions in §4 (Theorem 4.9), being based only on the Weak Coherence Theorem 1.2 combined with a method of *cuboid induction on dimension*; then they yield $H^1(\Omega, \mathcal{O}) = H^1(\Omega, \mathcal{I}\langle S \rangle) = 0$ for a holomorphically convex domain Ω (Lemma 4.19), which suffices to derive Oka's Heftungslemma or Grauert's finiteness theorem for \mathcal{O} (resp. and $\mathcal{I}\langle S \rangle$) on a strongly pseudoconvex domain (Theorem 4.21), and hence the solution of Levi's Problem on domains in \mathbf{C}^n (resp. unramified Riemann domains over \mathbf{C}^n).

2 Proof of Theorem 1.2

- (i) We take an arbitrary point $a \in \Omega$.

Case of $a \notin S$: Since S is closed, there is a neighborhood $U \subset \Omega$ of a with $U \cap S = \emptyset$. Then,

$$\mathcal{I}\langle S \rangle_x = \mathcal{O}_x = 1 \cdot \mathcal{O}_x, \quad \forall x \in U,$$

and therefore, $\{1\}$ is a finite generator system of $\mathcal{I}\langle S \rangle_x$ on U .

Case of $a \in S$: There is a holomorphic local coordinate neighborhood U of a with $z = (z_1, \dots, z_n)$ such that

$$(2.1) \quad \begin{aligned} a &= (0, \dots, 0) \in U = \text{P}\Delta(0; (r_j)), \\ S \cap U &= \{z = (z_j) \in U : z_1 = \dots = z_q = 0\} \quad (1 \leq q \leq n), \end{aligned}$$

¹⁾ A complex submanifold is not necessarily connected in this paper.

where $P\Delta(0; (r_j))$ denotes a polydisk with center at 0. Let $\underline{f}_b \in \mathcal{S}\langle S \rangle_b$ ($b \in U \cap S$) be any element. With the coordinate system (z_j) we write $b = (b_j) = (0, \dots, 0, b_{q+1}, \dots, b_n)$. The function f is represented by a unique power series expansion, $f(z) = \sum_{\nu \in \mathbf{Z}_+^n} c_\nu (z - b)^\nu$, which decomposes to

$$\begin{aligned} f(z) &= \sum_{\nu=(\nu_1, \nu') \in \mathbf{Z}_+^n, \nu_1 > 0} c_\nu (z - b)^\nu + \sum_{\nu=(\nu_1, \nu') \in \mathbf{Z}_+^n, \nu_1 = 0} c_\nu (z - b)^\nu \\ &= \left(\sum_{\nu=(\nu_1, \nu') \in \mathbf{Z}_+^n, \nu_1 > 0} c_\nu z_1^{\nu_1-1} (z' - b')^{\nu'} \right) z_1 + \sum_{\nu' \in \mathbf{Z}_+^{n-1}} c_{0\nu'} (z' - b')^{\nu'}. \end{aligned}$$

Here we put $\nu' = (\nu_2, \dots, \nu_n)$, $z' = (z_2, \dots, z_n)$, and $b' = (b_2, \dots, b_n)$. Setting

$$\begin{aligned} h_1(z_1, z') &= \left(\sum_{\nu=(\nu_1, \nu') \in \mathbf{Z}_+^n, \nu_1 > 0} c_\nu z_1^{\nu_1-1} (z' - b')^{\nu'} \right), \\ g_1(z') &= \sum_{\nu' \in \mathbf{Z}_+^{n-1}} c_{0\nu'} (z' - b')^{\nu'}, \end{aligned}$$

we have

$$(2.2) \quad f(z_1, z') = h_1(z_1, z') \cdot z_1 + g_1(z').$$

For $g_1(z')$ we apply a similar decomposition with respect to variable z_2 , so that

$$g_1(z') = h_2 \cdot z_2 + g_2(z''), \quad z'' = (z_3, \dots, z_n).$$

Repeating this process, we get

$$f(z) = \sum_{j=1}^q h_j(z) \cdot z_j + g_q(z_{q+1}, \dots, z_n).$$

If $z_1 = \dots = z_q = 0$, then $f(z) = 0$, and so $g_q(z_{q+1}, \dots, z_n) = 0$. Therefore,

$$f(z) = \sum_{j=1}^q h_j(z) \cdot z_j.$$

Thus,

$$(2.3) \quad \mathcal{S}\langle S \rangle|_U = \sum_{j=1}^q \mathcal{O}_U \cdot \underline{z}_j.$$

(ii) We begin with the following lemma:

Lemma 2.4. *With the natural complex coordinate system $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ we consider a relation sheaf \mathcal{R}_p ($1 \leq p \leq n$) defined by*

$$(2.5) \quad \underline{f}_{1_z} \underline{z}_{1_z} + \dots + \underline{f}_{p_z} \underline{z}_{p_z} = 0, \quad \underline{f}_{j_z} \in \mathcal{O}_z.$$

Then \mathcal{R}_p is finitely generated on \mathbf{C}^n by

$$(2.6) \quad T_{ij} = (0, \dots, 0, -\underline{z}_j, 0, \dots, 0, \underline{z}_i, 0, \dots, 0), \quad 1 \leq i < j \leq p.$$

We call T_{ij} ($1 \leq i < j \leq p$) of (2.6) the *trivial solutions* of (2.5) or of \mathcal{R}_p . In the case of $p = 1$, we set the trivial solution to be 0 as a convention.

Proof of Lemma 2.4: We use induction on $p \geq 1$. The case of $p = 1$ is clear.

Assuming that the case of $p - 1$ ($p \geq 2$) holds, we consider the case of p . Set

$$\Sigma = \{(z_1, \dots, z_n) : z_1 = \dots = z_p = 0\},$$

and let $a \in \mathbf{C}^n$ be an arbitrary point. If $a = (a_j) \notin \Sigma$, there is an $a_j \neq 0$ ($1 \leq j \leq p$), to say, $a_1 \neq 0$. In a neighborhood V of a , $z_1 \neq 0$. Then, (2.5) is solvable with respect to \underline{f}_{1_z} :

$$\underline{f}_{1_z} = -\underline{f}_{2_z} \frac{\underline{z}_{2_z}}{\underline{z}_{1_z}} - \dots - \underline{f}_{p_z} \frac{\underline{z}_{p_z}}{\underline{z}_{1_z}}, \quad \forall \underline{f}_{j_z} \in \mathcal{O}_z \ (2 \leq j \leq p), \ z \in V.$$

It follows that with $z \in V$,

$$(2.7) \quad \begin{aligned} (\underline{f}_{j_z}) &= \left(-\sum_{j=2}^p \underline{f}_{j_z} \frac{\underline{z}_{j_z}}{\underline{z}_{1_z}}, \underline{f}_{2_z}, \dots, \underline{f}_{p_z} \right) \\ &= \sum_{j=2}^p \frac{\underline{f}_{j_z}}{\underline{z}_{1_z}} \cdot \left(-\underline{z}_j, 0, \dots, 0, \underline{z}_1, 0, \dots, 0 \right) \\ &= \sum_{j=2}^p -\frac{\underline{f}_{j_z}}{\underline{z}_{1_z}} \cdot T_{1j}(z) \in \sum_{j=2}^p \mathcal{O}_z \cdot T_{1j}(z). \end{aligned}$$

Therefore, \mathcal{R}_p is generated by the trivial solutions $\{T_{1j}\}_{2 \leq j \leq p}$ on V .

If $a \in \Sigma$, we decompose an element $(\underline{f}_{j_a}) \in \mathcal{R}_{p_a}$ in a polydisk neighborhood U of a as in (2.2):

$$f_j(z_1, z') = h_j(z_1, z')z_1 + g_j(z'), \quad z' = (z_2, \dots, z_n), \ 1 \leq j \leq p.$$

For $z \in U$ one gets

$$(2.8) \quad \begin{aligned} (\underline{f}_{j_z}) - \sum_{j=2}^p \underline{h}_{j_z} T_{1j}(z) &= \left(\underline{g}_{1_z} + \sum_{j=1}^p \underline{h}_{j_z} \underline{z}_{j_z}, \underline{g}_{2_z}, \dots, \underline{g}_{p_z} \right) \\ &= (\underline{g}'_{1_z}, \underline{g}_{2_z}, \dots, \underline{g}_{p_z}). \end{aligned}$$

Here, $\underline{g}'_1 = \underline{g}_1 + \sum_{j=1}^p \underline{h}_{j_z} \underline{z}_{j_z}$. Since $(\underline{g}'_1, \underline{g}_{2_z}, \dots, \underline{g}_{p_z}) \in \mathcal{R}_{p_z}$,

$$\underline{g}'_1 \underline{z}_{1_z} + \underline{g}_{2_z} \underline{z}_{2_z} + \dots + \underline{g}_{p_z} \underline{z}_{p_z} = 0.$$

The second term and so forth of the right-hand side of the equation above do not contain variable z_1 , and so $\underline{g}'_{1_z} = 0$ is deduced. Thus,

$$\underline{g}_{2_z} \underline{z}_{2_z} + \dots + \underline{g}_{p_z} \underline{z}_{p_z} = 0.$$

This is the case of $p-1$ after changing the indices of variables. Therefore, the induction hypothesis implies that $(0, \underline{g}_{2_z}, \dots, \underline{g}_{p_z})$ is represented as a linear sum of $T_{ij}(z)$, $2 \leq i < j \leq p$, with coefficients in \mathcal{O}_z . Combining this with (2.8), we see that (\underline{f}_{j_z}) is represented as a linear sum of $T_{ij}(z)$, $1 \leq i < j \leq p$, with coefficients in \mathcal{O}_z . \triangle

Continued proof of (ii): Set $\mathcal{R} = \mathcal{R}(\underline{\sigma}_1, \dots, \underline{\sigma}_N)$. We consider the relation

$$(2.9) \quad \underline{f}_{1_z} \underline{\sigma}_{1_z} + \dots + \underline{f}_{N_z} \underline{\sigma}_{N_z} = 0, \quad \underline{f}_{j_z} \in \mathcal{O}_z.$$

We set the trivial solutions of this equation as follows:

$$\tau_{ij} = (\dots, \overset{i\text{-th}}{-\underline{\sigma}_j}, \dots, \overset{j\text{-th}}{\underline{\sigma}_i}, \dots), \quad 1 \leq i < j \leq N.$$

We take an arbitrary point $a \in \Omega$. If $a \notin S$, then some $\sigma_j(a) \neq 0$, to say, $\sigma_1(a) \neq 0$. As in (2.7), one sees that \mathcal{R} is generated by $\{\tau_{1j}\}_{j=2}^N$ on a neighborhood of a .

If $a \in S$, we take a holomorphic local coordinate system $z = (z_1, \dots, z_n)$ in a polydisk neighborhood $P\Delta$ as in (2.1):

$$a = (0, \dots, 0),$$

$$S \cap P\Delta = \{(z_1, \dots, z_n) \in P\Delta : z_1 = \dots = z_q = 0\} \quad (1 \leq q \leq n).$$

It follows from (2.3) and the assumption that

$$\mathcal{J}\langle S \rangle|_{P\Delta} = \sum_{j=1}^q \mathcal{O}_{P\Delta} \cdot \underline{z}_j = \sum_{j=1}^N \mathcal{O}_{P\Delta} \cdot \underline{\sigma}_j|_{P\Delta}.$$

Thus, we may assume without loss of generality that

$$\sigma_j = z_j, \quad 1 \leq j \leq q \text{ (on } P\Delta),$$

$$\sigma_i = \sum_{j=1}^q a_{ij} z_j, \quad a_{ij} \in \mathcal{O}(P\Delta), \quad q+1 \leq i \leq N \text{ (on } P\Delta).$$

Set

$$(2.10) \quad \phi_i = (-\underline{a}_{i1}, \dots, -\underline{a}_{iq}, 0, \dots, 0, \overset{i\text{-th}}{1}, 0, \dots, 0) \in \Gamma(P\Delta, \mathcal{R}), \quad q+1 \leq i \leq N.$$

We deduce from (2.9) with $z \in P\Delta$ that

$$(2.11) \quad \left(\underline{f}_1 + \sum_{i=q+1}^N \underline{f}_i \underline{a}_{i1} \right) \underline{z}_1 + \cdots + \left(\underline{f}_q + \sum_{i=q+1}^N \underline{f}_i \underline{a}_{iq} \right) \underline{z}_q = 0.$$

By Lemma 2.4,

$$\left(\underline{f}_1 + \sum_{i=q+1}^N \underline{f}_i \underline{a}_{i1}, \dots, \underline{f}_q + \sum_{i=q+1}^N \underline{f}_i \underline{a}_{iq}, 0, \dots, 0 \right)$$

is a linear sum of τ_{jk} , $1 \leq j < k \leq q$, with coefficients in \mathcal{O}_z . Therefore there are $\underline{b}_{jk} \in \mathcal{O}_z$, $1 \leq j < k \leq q$, such that

$$(2.12) \quad \sum_{1 \leq j < k \leq q} \underline{b}_{jk} \tau_{jk}(z) = \left(\underline{f}_1 + \sum_{i=q+1}^N \underline{f}_i \underline{a}_{i1}, \dots, \underline{f}_q + \sum_{i=q+1}^N \underline{f}_i \underline{a}_{iq}, 0, \dots, 0 \right) \\ = \left(\underline{f}_1, \dots, \underline{f}_q, 0, \dots, 0 \right) + \sum_{i=q+1}^N \underline{f}_i \left(\underline{a}_{i1}, \dots, \underline{a}_{iq}, 0, \dots, 0 \right).$$

By making use of (2.10) we get

$$(2.13) \quad \left(\underline{f}_1, \dots, \underline{f}_q, \dots, \underline{f}_N \right) = \sum_{1 \leq j < k \leq q} \underline{b}_{jk} \tau_{jk}(z) + \sum_{i=q+1}^N \underline{f}_i \phi_i(z).$$

Thus, \mathcal{R} is generated on $P\Delta$ by

$$(2.14) \quad \tau_{jk}, \phi_i, \quad 1 \leq j < k \leq q, \quad q+1 \leq i \leq N.$$

This finishes the proof. \square

Remark 2.15. (i) In the Weak Coherence Theorem 1.2 it is the point to assume that $\{\underline{\sigma}_j\}_{j=1}^N$ is a generator system of $\mathcal{S}\langle S \rangle$; otherwise, the proof above does not work even if S is non-singular.

(ii) It is an advantage of the above method to the general First Coherence Theorem of Oka that we have an explicit system of generators (2.14).

(iii) To show the local finiteness of the relation sheaf of the generators (2.14) it is necessary to prepare Oka's First Coherent Theorem in general form proved with Weierstrass' Preparation Theorem.

3 Oka's Jôku-Ikô

The term “Jôku-Ikô” was used by K. Oka since he wrote the first paper in series in 1936 ([15]—[24]), and means a principle to transform a difficult problem into higher dimensional domains of a simple shape such as polydisks, and to solve it. He retained this principle all through the series of papers from I to IX ([15]—[24]); The aim of the present section is to prove Oka's Jôku-Ikô Lemma 3.10 only by making use of Theorem 1.2 combined with Cousin's integral (3.7). The technics may essentially be similar to those in some references, e.g., Nishino [10] and Noguchi [12], but they are not in a suitable form for our purpose.

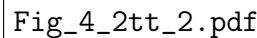
3.1 Syzygy for non-singular geometric ideal sheaves

We begin with:

Definition 3.1. A cuboid E is a bounded open or closed subset of \mathbf{C}^n with the boundary parallel to the real and imaginary axes of $z = (z_1, \dots, z_n) \in \mathbf{C}^n$. In the case of $n = 1$, E is called a rectangle. When E is a closed cuboid, we allow the widths of some edges to degenerate to 0, and call the number of edges of E of positive widths the dimension of E , denoted by $\dim E$.

Let $\Omega \subset \mathbf{C}^n = \mathbf{C}^{n-1} \times \mathbf{C}$ be a domain and let $E', E'' \Subset \Omega$ be two closed cuboids as follows: There are a closed cuboid $F \Subset \mathbf{C}^{n-1}$ and two adjacent closed rectangles $E'_n, E''_n \Subset \mathbf{C}$ sharing a side ℓ , and

$$(3.2) \quad E' = F \times E'_n, \quad E'' = F \times E''_n, \quad \ell = E'_n \cap E''_n.$$



Fig_4_2tt_2.pdf

Figure 1: Adjacent closed cuboids

We now recall:

Lemma 3.3 (Cartan's Merging Lemma). *Let $E', E'' \Subset \Omega$ be adjacent closed cuboids as in (3.2), and let \mathcal{F} be an analytic sheaf on Ω . Let $\{\sigma'_j \in \Gamma(U', \mathcal{F}) : 1 \leq j \leq p'\}$ (resp. $\{\sigma''_k \in \Gamma(U'', \mathcal{F}), 1 \leq k \leq p''\}$) be a finite generator system of \mathcal{F} on E' (resp. E'').²⁾*

²⁾ This means that they are defined so in some neighborhoods of E' and E'' , respectively; this expression is the same through the paper.

Moreover, assume that there are holomorphic functions $a_{jk}, b_{kj} \in \mathcal{O}(E' \cap E'')$, $1 \leq j \leq p'$, $1 \leq k \leq p''$, such that

$$\sigma'_j = \sum_{k=1}^{p''} \underline{a_{jk}} \cdot \sigma''_k, \quad \sigma''_k = \sum_{j=1}^{p'} \underline{b_{kj}} \cdot \sigma'_j \quad (\text{on } E' \cap E'').$$

Then, there exists a merged finite generator system $\{\sigma_l \in \Gamma(E' \cup E''), \mathcal{F}\} : 1 \leq l \leq p' + p''\}$ on $E' \cup E''$.

The proof is done by Cartan's matrix decomposition lemma³⁾, which does not involve the coherence property (cf., e.g., [7], [10], [12]).

Lemma 3.4 (Oka's Syzygy). *Let $E \Subset \mathbf{C}^n$ be a closed cuboid.*

- (i) *Every locally finite analytic sheaf \mathcal{F} defined on E (i.e., in a neighborhood of E) has a finite generator system on E .*
- (ii) *Let \mathcal{F} be an analytic sheaf on E with a finite generator system $\{\sigma_j\}_{1 \leq j \leq N}$ on E such that the relation sheaf $\mathcal{R}(\sigma_1, \dots, \sigma_N)$ is locally finite.*

Then for every section $\sigma \in \Gamma(E, \mathcal{F})$ there are holomorphic functions $a_j \in \mathcal{O}(E)$, $1 \leq j \leq N$, such that

$$(3.5) \quad \sigma = \sum_{j=1}^N \underline{a_j} \cdot \sigma_j \quad (\text{on } E).$$

Proof. The proof is carried out in the same way as in [10], or [12] Lemma 4.3.7 except for the use of the vanishing $H^1(U, \mathcal{O}) = 0$ for a convex cylinder domain $U \subset \mathbf{C}^n$, which we replace by Cousin's integral as follows. Suppose that E is a closed cuboid such that

$$(3.6) \quad E = F \times \{z_n : |\Re z_n| \leq T, |\Im z_n| \leq \theta\}, \quad T > 0, \theta \geq 0.$$

Set $E_0 = F \times \{z_n : \Re z_n = 0, |\Im z_n| \leq \theta\}$, and let $\varphi(z', z_n) \in \mathcal{O}(E_0)$. Then there is a small $\delta > 0$ such that $\varphi(z', z_n)$ is defined on

$$F \times \{z_n : |\Re z_n| \leq \delta, |\Im z_n| \leq \theta + \delta\}.$$

Set

$$\begin{aligned} \ell &= \{z_n : \Re z_n = 0, -\theta - \delta \leq \Im z_n \leq \theta + \delta\}, \\ E_1 &= F \times \{z_n : -T \leq \Re z_n \leq \delta, |\Im z_n| \leq \theta\}, \\ E_2 &= F \times \{z_n : -\delta \leq \Re z_n \leq T, |\Im z_n| \leq \theta\}, \end{aligned}$$

³⁾ A rather simplified proof of this lemma may be found in [12], Added at galley-proof.

where ℓ is positively oriented as $\Im z_n$ increases. We define Cousin's integral of $\varphi(z', z_n)$ along ℓ by

$$\Phi(z', z_n) = \frac{1}{2\pi i} \int_{\ell} \frac{\varphi(z', \zeta_n)}{\zeta_n - z_n} d\zeta_n.$$

Then $\Phi(z', z_n)$ is holomorphic on $(E_1 \cup E_2) \setminus (F \times \ell)$. After analytic continuations we obtain $\Phi_j(z', z_n) \in \mathcal{O}(E_j)$ ($j = 1, 2$) satisfying

$$(3.7) \quad \Phi_1(z', z_n) - \Phi_2(z', z_n) = \varphi(z', z_n), \quad (z', z_n) \in E_1 \cap E_2.$$

We call this the *Cousin decomposition* of $\varphi(z', z_n)$.

The rest is the same as in the proof of [12] Lemma 4.3.7. □

By the Weak Coherence Theorem 1.2 and Lemma 3.4 we have:

Theorem 3.8 (Syzygy for $\mathcal{J}\langle S \rangle$). *Let S be a complex submanifold of a neighborhood of a closed cuboid $E \subset \mathbf{C}^n$.*

- (i) $\mathcal{J}\langle S \rangle$ has a finite generator system on E .
- (ii) Let $\{\sigma_j\}_{1 \leq j \leq N}$ be a finite generator system of $\mathcal{J}\langle S \rangle$ on E with $\sigma_j \in \mathcal{O}(E)$. Then for every $\underline{\sigma} \in \Gamma(E, \mathcal{J}\langle S \rangle)$ ($\sigma \in \mathcal{O}(E)$) there are holomorphic functions $a_j \in \mathcal{O}(E)$, $1 \leq j \leq N$, such that

$$(3.9) \quad \sigma = \sum_{j=1}^N a_j \cdot \sigma_j \quad (\text{on } E).$$

3.2 Oka's Jôku-Ikô

Let P be an open cuboid in \mathbf{C}^n , and let $S \subset P$ be a complex submanifold. The following is fundamental in the Oka theory.

Lemma 3.10 (Oka's Jôku-Ikô). *Let $E \Subset P$ be a closed cuboid. Then for every holomorphic function g on $E \cap S$ ($\Subset S$)⁴⁾ there exists a "solution" $G \in \mathcal{O}(E)$ satisfying*

$$G|_{E \cap S} = g|_{E \cap S}.$$

*Here, the equality holds in a neighborhood of $E \cap S$ in S .*⁵⁾

Proof. Notice that in the case of $E \cap S = \emptyset$, G can be any holomorphic function on E , and the statement is true. We use induction on $\dim E$.

- (a) Case of $\dim E = 0$: Since E consists of one point, the assertion is clear.

⁴⁾With this writing we mean that g is a holomorphic function in a neighborhood V of $E \cap S$ in S . The notation will be used in sequel.

⁵⁾ The formulation of this lemma and the proof below should be new.

(b) Case of $\dim E = \nu$ ($\nu \geq 1$) with the induction hypothesis that the case of $\dim E = \nu - 1$ is true: By Theorem 3.8 (i) there is a finite generator system $\{\underline{\sigma}_j\}_{j=1}^N$ of $\mathcal{J}\langle S \rangle$ on a neighborhood $W(\subset P)$ of E with $\sigma_j \in \mathcal{O}(W)$.

We may assume that E is taken as in (3.6). We set

$$(3.11) \quad E_t = \{z = (z', z_n) \in E : \Re z_n = t\}, \quad -T \leq t \leq T.$$

Since $\dim E_t = \nu - 1$, the induction hypothesis implies that there is a solution $G_t \in \mathcal{O}(E_t)$ satisfying $G_t|_{S \cap E_t} = g|_{S \cap E_t}$. By the Heine–Borel Theorem there is a finite partition

$$(3.12) \quad \begin{aligned} -T = t_0 < t_1 < \cdots < t_L = T, \\ E_\alpha := \{z = (z', z_n) \in E : t_{\alpha-1} \leq \Re z_n \leq t_\alpha\}, \quad 1 \leq \alpha \leq L, \end{aligned}$$

such that there are solutions $G_\alpha \in \mathcal{O}(E_\alpha)$ satisfying

$$G_\alpha|_{S \cap E_\alpha} = g|_{S \cap E_\alpha}.$$

Therefore, $\underline{G_{\alpha+1} - G_\alpha} \in \Gamma(E_\alpha \cap E_{\alpha+1}, \mathcal{J}\langle S \rangle)$. It follows from Theorem 3.8 (ii) that there are $a_{\alpha j} \in \mathcal{O}(E_\alpha \cap E_{\alpha+1})$ ($1 \leq j \leq N$) satisfying

$$(3.13) \quad G_{\alpha+1} - G_\alpha = \sum_{j=1}^N a_{\alpha j} \sigma_j \quad (\text{on } E_\alpha \cap E_{\alpha+1}).$$

By the Cousin decomposition (3.7) of $a_{\alpha j}$ we write

$$(3.14) \quad a_{\alpha j} = b_{\alpha j} - b_{\alpha+1 j} \quad (\text{on } E_\alpha \cap E_{\alpha+1}), \quad b_{\alpha j} \in \mathcal{O}(E_\alpha), \quad b_{\alpha+1 j} \in \mathcal{O}(E_{\alpha+1}).$$

Then,

$$(3.15) \quad G_\alpha + \sum_{j=1}^N b_{\alpha j} \sigma_j = G_{\alpha+1} + \sum_{j=1}^N b_{\alpha+1 j} \sigma_j \quad (\text{on } E_\alpha \cap E_{\alpha+1}).$$

Thus this yields a solution $H_{\alpha+1}$ on $E_\alpha \cup E_{\alpha+1}$; for this procedure we say that we merge the solutions G_α and $G_{\alpha+1}$ to obtain a solution $H_{\alpha+1}$ on $E_\alpha \cup E_{\alpha+1}$.

Starting from $\alpha = 1$, we merge G_1 and G_2 to obtain a solution H_2 on $E_1 \cup E_2$. We then merge H_2 and G_3 to obtain a solution H_3 on $E_1 \cup E_2 \cup E_3$. Repeating this procedure up to $\alpha = L - 1$, we obtain a solution H_L on $E = \bigcup_{\alpha=1}^L E_\alpha$, and set $G = H_L$: This finishes the proof of Lemma 3.10. \square

Remark 3.16. We call the above induction argument *cuboid induction on dimension*, which will be used furthermore in the sequel.

It is well-known that Oka's Jôku-Ikô Lemma 3.10 immediately implies (cf., e.g., [12] Lemma 4.4.17):

Theorem 3.17 (Runge–Weil–Oka Approximation). *Let $\Delta \Subset \Omega$ be an analytic polyhedron of a domain $\Omega (\subset \mathbb{C}^n)$. Then every holomorphic function on the closure $\bar{\Delta}$ is uniformly approximated on $\bar{\Delta}$ by elements of $\mathcal{O}(\Omega)$.*

4 Cousin I, II, $\bar{\partial}$, Extension and Levi's Problems

The aim of this section is to show how the result obtained in the previous section is applied to solve the titled problems.

4.1 Cousin I, II, and $\bar{\partial}$ -Equation

We will give one unified proof to all of the three problems. We recall them: Let $\Omega \subset \mathbf{C}^n$ be a domain, let $\Omega = \bigcup_{\alpha \in \Lambda} U_\alpha$ be an open covering, and let $\mathcal{M}(U_\alpha)$ denote the set of all meromorphic functions in U_α .

- I (**Cousin I**) For given $f_\alpha \in \mathcal{M}(U_\alpha)$ ($\alpha \in \Lambda$) satisfying $f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)$ (Cousin-I data), find $F \in \mathcal{M}(\Omega)$ (*solution*) with $F|_{U_\alpha} - f_\alpha \in \mathcal{O}(U_\alpha)$ for all $\alpha \in \Lambda$.
- II (**Cousin II**) Here we assume that U_α are simply-connected. Let $f_\alpha \in \mathcal{M}^*(U_\alpha)$ ($\alpha \in \Lambda$) be locally non-zero meromorphic functions satisfying
 - (a) $f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ (nowhere vanishing holomorphic functions) (Cousin-II data),
 - (b) (Topological condition) there are nowhere vanishing continuous functions $\psi_\alpha \in \mathcal{C}^*(U_\alpha)$ with $\psi_\alpha/\psi_\beta = f_\beta/f_\alpha$ on $U_\alpha \cap U_\beta$.

Find $F \in \mathcal{M}^*(\Omega)$ with $F|_{U_\alpha}/f_\alpha \in \mathcal{O}^*(U_\alpha)$ for all $\alpha \in \Lambda$. Equivalently, find a continuous function $\Psi \in \mathcal{C}(\Omega)$ (*solution*) with $\Psi|_{U_\alpha} - \log \psi_\alpha \in \mathcal{O}(U_\alpha)$ for all $\alpha \in \Lambda$.

- III (**$\bar{\partial}$ -Equation**) For a given C^∞ -(0,1)-form u on Ω with $\bar{\partial}u = 0$, find a C^∞ -function g on Ω with $\bar{\partial}g = u$.

Locally, by Dolbeault's lemma, there is a solution f of this problem in a neighborhood of a point of Ω . Thus, there is an open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of Ω and C^∞ -functions g_α on U_α such that $\bar{\partial}g_\alpha = u|_{U_\alpha}$. Then, the present problem is equivalent to find a C^∞ -function G (*solution*) on Ω with $G|_{U_\alpha} - g_\alpha \in \mathcal{O}(U_\alpha)$ for all $\alpha \in \Lambda$.

Convention. For a unified treatment for the above problems, we introduce an “*argument* χ ” representing one of I—III above: Problem- χ means one of Problems I—III above, and a χ -solution means a *solution* of the corresponding Problem- χ .

Remark 4.1. If Ψ is so obtained in Cousin-II Problem above, then $F_1 = f_\alpha e^{\psi_\alpha - \Psi} \in \mathcal{M}^*(\Omega)$ satisfies the required property for F . Then we have a homotopy,

$$F_t = f_\alpha e^{\log \psi_\alpha - t\Psi}, \quad 0 \leq t \leq 1,$$

from the topologically assumed function $F_0 (= f_\alpha \psi_\alpha)$ to an aimed analytic (meromorphic) function F_1 .

Remark 4.2. The common property of Problem- χ that we will use is the following: If f and f' are two solutions of Problem- χ on an open set U in general, then $f - f' \in \mathcal{O}(U)$.

We begin with:

Lemma 4.3. *Let P be an open cuboid in \mathbf{C}^n and let S be a complex submanifold of P . We consider Problem- χ defined on S . Let $E \Subset P$ be a closed cuboid. Then there is a χ -solution on $E \cap S (\Subset S)$.⁶⁾*

Proof. We use cuboid induction on dimension.

(a) Case of $\dim E = 0$: It is clear by definition.

(b) Case of $\dim E = \nu (\nu \geq 1)$ with the induction hypothesis that the case of $\dim E = \nu - 1$ holds: Without loss of generality we may assume that E is given as in (3.6), and let E_t be as in (3.11). Since $\dim E_t = \nu - 1$, the induction hypothesis implies the existence of a χ -solution Φ_t on $E_t \cap S (\Subset S)$. Then, by the Heine-Borel Theorem there are a partition of $[-T, T]$, E_α ($1 \leq \alpha \leq L$) as in (3.12), and χ -solutions Φ_α on $E_\alpha \cap S (\Subset S)$.

If $E_\alpha \cap E_{\alpha+1} \cap S \neq \emptyset$, we say that E_α and $E_{\alpha+1}$ is pairwise connected on S . It is sufficient to prove the existence of a χ -solution for each maximal sequence of E_α pairwise connected on S ,

$$(4.4) \quad E_{\alpha_0} \cup E_{\alpha_0+1} \cup \cdots \cup E_{\alpha_1}.$$

For simplicity we suppose that $\alpha_0 = 1$. It follows from Remark 4.2 that for $1 \leq \alpha \leq \alpha_1$

$$(4.5) \quad \Phi_{\alpha+1} - \Phi_\alpha \in \Gamma(E_\alpha \cap E_{\alpha+1} \cap S, \mathcal{O}_S).$$

By Oka's Jôku-Ikô Lemma 3.10, there is a holomorphic function $H_\alpha \in \mathcal{O}(E_\alpha \cap E_{\alpha+1})$ such that

$$(4.6) \quad H_\alpha|_{E_\alpha \cap E_{\alpha+1} \cap S} = \Phi_{\alpha+1} - \Phi_\alpha.$$

By the Cousin decomposition of H_α as in (3.7) we have $\tilde{H}_\alpha \in \mathcal{O}(E_\alpha)$ and $\tilde{H}_{\alpha+1} \in \mathcal{O}(E_{\alpha+1})$ such that

$$(4.7) \quad H_\alpha = \tilde{H}_\alpha - \tilde{H}_{\alpha+1} \quad (\text{on } E_\alpha \cap E_{\alpha+1}).$$

We infer from (4.7) and (4.13) that

$$(4.8) \quad \Phi_\alpha + \tilde{H}_\alpha|_{E_\alpha \cap S} = \Phi_{\alpha+1} + \tilde{H}_{\alpha+1}|_{E_{\alpha+1} \cap S} \quad \text{on } E_\alpha \cap E_{\alpha+1} \cap S (\Subset S).$$

Note that $\Phi_\alpha + \tilde{H}_\alpha|_{E_\alpha \cap S}$ (resp. $\Phi_{\alpha+1} + \tilde{H}_{\alpha+1}|_{E_{\alpha+1} \cap S}$) is a χ -solution on $E_\alpha \cap S (\Subset S)$ (resp. $E_{\alpha+1} \cap S (\Subset S)$). Thus, from (4.8) we obtain a merged χ -solution $\Psi_{\alpha+1}$ on $(E_\alpha \cup E_{\alpha+1}) \cap S (\Subset S)$ from Φ_α and $\Phi_{\alpha+1}$.

⁶⁾ Cf. footnote 4) at p. 9.

Now, from Φ_1 and Φ_2 we obtain a merged χ -solution Ψ_2 on $(E_1 \cup E_2) \cap S(\Subset S)$. We then obtain a merged χ -solution Ψ_3 on $(E_1 \cup E_2 \cup E_3) \cap S(\Subset S)$ from Ψ_2 and Φ_3 , and so on; we obtain a χ -solution on $(\bigcup_{\alpha=1}^{\alpha_1} E_\alpha) \cap S(\Subset S)$. \square

Theorem 4.9. *Let Ω be a holomorphically convex domain (equivalently, a domain of holomorphy). Then Problem- χ on Ω has a χ -solution on Ω .*

Proof. We take an increasing sequence of analytic polyhedra of Ω ,

$$(4.10) \quad \Delta_1 \Subset \Delta_2 \Subset \Delta_3 \Subset \cdots, \quad \bigcup_{\nu=1}^{\infty} \Delta_\nu = \Omega.$$

For each ν we let $\phi_\nu : \bar{\Delta}_\nu \rightarrow \overline{P\Delta}_\nu$ be the Oka map (a holomorphic proper embedding) of $\bar{\Delta}_\nu$ into a closed polydisk $\overline{P\Delta}_\nu$, which extends from a neighborhood U_ν of $\bar{\Delta}_\nu$ into a polydisk, biholomorphic to an open cuboid $P_\nu (\ni \bar{P\Delta}_\nu)$. Then, the image $\phi_\nu(U_\nu)$ is a complex submanifold of P_ν . We identify U_ν with the image $\phi_\nu(U_\nu)$.

By Lemma 4.3 there is a χ -solution G_ν on every $\bar{\Delta}_\nu$. Put $F_1 = G_1$ on $\bar{\Delta}_1$. Suppose that χ -solutions F_ν on $\bar{\Delta}_\nu$, $1 \leq \nu \leq \mu$, are determined so that

$$(4.11) \quad \|F_{\nu+1} - F_\nu\|_{\bar{\Delta}_\nu} < \frac{1}{2^\nu}, \quad 1 \leq \nu \leq \mu.$$

Let $G_{\mu+1}$ be a χ -solution on $\bar{\Delta}_{\mu+1}$. Since $G_{\mu+1}|_{\bar{\Delta}_\mu} - F_\mu \in \mathcal{O}(\bar{\Delta}_\mu)$, by Theorem 3.17 there is an element $h_{\mu+1} \in \mathcal{O}(\bar{\Delta}_{\mu+1})$ with

$$\|G_{\mu+1}|_{\bar{\Delta}_\mu} - F_\mu - h_{\mu+1}\|_{\bar{\Delta}_\mu} < \frac{1}{2^{\mu+1}}.$$

Setting $F_{\mu+1} = G_{\mu+1} - h_{\mu+1}$, we see that (4.11) holds up to $\mu + 1$. Inductively, we have χ -solutions F_ν on $\bar{\Delta}_\nu$ satisfying (4.11), and the series

$$F = F_\mu + \sum_{\nu=\mu}^{\infty} (F_{\nu+1} - F_\nu)$$

converges locally uniformly and the limit gives rise to a χ -solution on Ω . \square

Remark 4.12. As easily seen, the above proof of Theorem 4.9 works on Stein manifolds.

4.2 Extension Problem

By means of the Weak Coherence Theorem 1.2 we consider the extension problem (interpolation problem) from a complex submanifold in a holomorphically convex domain.

Theorem 4.13. *Let $\Omega \subset \mathbf{C}^n$ be a holomorphically convex domain and let $S \subset \Omega$ be a complex submanifold. Then the restriction map*

$$F \in \mathcal{O}(\Omega) \rightarrow F|_S \in \mathcal{O}(S)$$

is a surjection.

Proof. We take analytic polyhedra $\Delta_\nu \Subset \Omega$ and Oka maps $\phi_\nu : \bar{\Delta}_\nu (\Subset U_\nu) \rightarrow \overline{P\Delta}_\nu (\Subset P_\nu)$ ($\nu = 1, 2, \dots$) as in the proof of Theorem 4.9. By Theorem 3.8 (i) there is a finite generator system $\{\underline{\sigma}_{\nu j}\}_{j=1}^{N_\nu}$ of $\mathcal{J}\langle S \cap P_\nu \rangle$ on each $\overline{P\Delta}_\nu (\Subset P_\nu)$, where U_ν is identified with $\phi_\nu(U_\nu)$.

Let $f \in \mathcal{O}(S)$ be any element. By Oka's Jôku-Ikô Lemma 3.10 there are $G_\nu \in \mathcal{O}(\overline{P\Delta}_\nu)$ with $G_\nu|_{\bar{\Delta}_\nu \cap S} = f|_{\bar{\Delta}_\nu \cap S}$ ($\nu = 1, 2, \dots$).

We set $F_1 = G_1|_{\bar{\Delta}_1}$. Suppose that $F_\nu \in \mathcal{O}(\bar{\Delta}_\nu)$, $1 \leq \nu \leq \mu$, are determined so that

$$(4.14) \quad F_\nu = f|_{\bar{\Delta}_\nu \cap S}, \quad \|F_{\nu+1} - F_\nu\|_{\overline{P\Delta}_\nu} < \frac{1}{2^\nu}, \quad 1 \leq \nu \leq \mu - 1.$$

For $\nu = \mu + 1$ we first note that $(G_{\mu+1}|_{\bar{\Delta}_\mu} - F_\mu)|_{\bar{\Delta}_\mu \cap S} = 0$. By Lemma 3.10 there is an element $H_\mu \in \mathcal{O}(\overline{P\Delta}_\mu)$ with $H_\mu|_{\bar{\Delta}_\mu} = G_{\mu+1}|_{\bar{\Delta}_\mu} - F_\mu$. Since $H_\mu \in \Gamma(\overline{P\Delta}_\mu, \mathcal{J}\langle S \rangle)$, by Theorem 3.8 (ii) there are $h_{\mu j} \in \mathcal{O}(\overline{P\Delta}_\mu)$, $1 \leq j \leq N_{\mu+1}$, such that

$$H_\mu = \sum_{j=1}^{N_{\mu+1}} h_{\mu j} \cdot \sigma_{\mu+1 j} \quad \text{on } \overline{P\Delta}_\mu.$$

Restricting this to $\bar{\Delta}_\nu$, we have

$$G_{\mu+1}|_{\bar{\Delta}_\mu} = F_\mu + \sum_{j=1}^{N_{\mu+1}} h_{\mu j} \cdot \sigma_{\mu+1 j}|_{\bar{\Delta}_\mu}.$$

Approximating $h_{\mu j}$ sufficiently close by $\tilde{h}_{\mu j} \in \mathcal{O}(\Omega)$ on $\bar{\Delta}_\mu$ (Theorem 3.17), and setting

$$F_{\mu+1} = G_{\mu+1} - \sum_{j=1}^{N_{\mu+1}} \tilde{h}_{\mu j} \cdot \sigma_{\mu+1 j} \in \mathcal{O}(\bar{\Delta}_{\mu+1}),$$

we have

$$F_{\mu+1}|_{\bar{\Delta}_{\mu+1} \cap S} = f|_{\bar{\Delta}_{\mu+1} \cap S}, \quad \|F_{\mu+1} - F_\mu\|_{\bar{\Delta}_\mu} < \frac{1}{2^\mu}.$$

Then the series

$$F = F_\mu + \sum_{\nu=\mu}^{\infty} (F_{\nu+1} - F_\nu)$$

converges locally uniformly to the limit $F \in \mathcal{O}(\Omega)$ with $F|_S = f$. □

Remark 4.15. The above proof of Theorem 4.13 works on Stein manifolds.

4.3 Levi's Problem

4.3.1 Oka's method

Notice that Oka's Jôku-Ikô Lemma 3.10 is sufficient to deduce *Oka's Heftungslemma* which, together with a method of an integral equation and the construction of a plurisubharmonic exhaustion on a pseudoconvex unramified Riemann domain over \mathbb{C}^n , implies

Levi's Problem (Hartogs' Inverse Problem) (cf. Oka [20], [21], [25], [24], Andreotti-Narasimhan [1], Nishino [10]):

Theorem 4.16 (Oka, 1941/42/43/53; cf. Remark 4.22). *Let Ω be a unramified Riemann domain over \mathbf{C}^n . If Ω is pseudoconvex, then Ω is a Stein manifold.*

4.3.2 Grauert's method

In 1958 H. Grauert [6] gave another proof of Theorem 4.16 by proving the finite dimensionality of the first cohomology of coherent sheaves which was inspired by the Cartan–Serre Theorem for coherent sheaves on compact analytic spaces.⁷⁾ We shall observe that the Weak Coherence Theorem 1.2 suffices for Grauert's method to prove Theorem 4.16.

We first recall Leray's theorem on Čech cohomologies $H^r(*, \mathcal{O}_*)$ in our restricted setting, which we will use only for $r = 1$:

Theorem 4.17. *Let $\mathcal{S} \rightarrow X$ be a sheaf of abelian groups over a complex manifold X . Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X . Let $r \in \mathbf{N}$ be a positive integer. Suppose that for all pairs $(p, q) \in \mathbf{N}^2$ with $1 \leq p + q \leq r$*

$$H^p(\text{supp } \sigma, \mathcal{S}) = 0, \quad \forall \sigma \in N_q(\mathcal{U}),$$

where $N_q(\mathcal{U})$ denotes the set of all q -simplices of \mathcal{U} . Then,

$$H^r(X, \mathcal{S}) \cong H^r(\mathcal{U}, \mathcal{S}).$$

We also recall:

Theorem 4.18 (Dolbeault). *Let X be a complex manifold and let $q \geq 0$. Then*

$$H^q(X, \mathcal{O}_X) \cong \{\omega : C^\infty\text{-}(0, q)\text{-form on } X, \bar{\partial}\omega = 0\} / \bar{\partial}\{\eta : C^\infty\text{-}(0, q-1)\text{-form on } X\}.$$

Lemma 4.19. *Let Ω be a holomorphically convex domain of \mathbf{C}^n and let $S \subset \Omega$ be a complex submanifold. Then we have:*

- (i) $H^1(\Omega, \mathcal{O}) = 0$.
- (ii) $H^1(\Omega, \mathcal{S}\langle S \rangle) = 0$.

Proof. (i) This follows from Theorem 4.9 with $\chi = \text{III}$ and Theorem 4.18.

(ii) We use the following exact sequence:

$$H^0(\Omega, \mathcal{O}) \xrightarrow{r} H^0(S, \mathcal{O}_S) \rightarrow H^1(\Omega, \mathcal{S}\langle S \rangle) \rightarrow H^1(\Omega, \mathcal{O}) = 0,$$

where r is the restriction map and (i) was used. By Theorem 4.13, r is surjective. Therefore, $H^1(\Omega, \mathcal{S}\langle S \rangle) = 0$. \square

Combining this with Theorem 4.17, we get

⁷⁾ Cf. the footnote of [6] p. 466. The proof relies on L. Schwartz's finiteness theorem, whose rather simple, short and complete proof is found in [4] and [12] pp. 313–315.

Lemma 4.20. *Let $S \subset X$ be a complex submanifold of a complex manifold X . Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X such that all U_α are biholomorphic to holomorphically convex domains of \mathbf{C}^n . Then,*

$$H^1(X, \mathcal{O}_X) \cong H^1(\mathcal{U}, \mathcal{O}_X), \quad H^1(X, \mathcal{I}\langle S \rangle) \cong H^1(\mathcal{U}, \mathcal{I}\langle S \rangle).$$

Then we can apply Grauert's bumping method [6] to prove:

Theorem 4.21 (Grauert). *Let $\Omega \Subset X$ be a relatively compact domain of a complex manifold X with strongly pseudoconvex boundary. Let S be a complex submanifold of X . Then the following holds:*

- (i) $\dim_{\mathbf{C}} H^1(\Omega, \mathcal{O}_\Omega) < \infty$,
- (ii) $\dim_{\mathbf{C}} H^1(\Omega, \mathcal{I}\langle S \rangle) < \infty$.

Then this finite dimensionality theorem implies Theorem 4.16, where (i) is sufficient for $\Omega \subset \mathbf{C}^n$, but moreover (ii) is necessary for unramified Riemann domains over \mathbf{C}^n (cf., [11], [12] Chap. 7).

Remark 4.22 (Historical comments; cf. also [12] Chap. 9 “On Coherence”). Oka's Theorem 4.16 was first proved for $\Omega \subset \mathbf{C}^2$ by Oka [20] (announcement) in 1941, and the full paper [21] was published in 1942 with a comment of the validity for $n \geq 3$.

In 1943 he proved Theorem 4.16 for unramified Riemann domains of general dimension ≥ 2 in a series of research reports of pp. 109 in total, sent to Teiji Takagi: The reports were written in Japanese and unpublished.⁸⁾ Oka remarked this fact twice in [23] and [24]. In the 1943 reports to Takagi he did not use Weierstrass' Preparation Theorem, but he was writing a primitive form of the notion of presheaves and non-reduced structures of analytic subsets; he later called the notion “*idéal de domaines indéterminés*” in [22] written in 1948. The key of Oka's proof of Theorem 4.16 was his “*Heftungslemma*”. In [20] and [21] he proved Heftungslemma by Weil's integral, but in 1943 ([25] no. 1) he replaced Weil's integral by simple Cauchy's integral by proving “Oka's Jôku-Ikô” for Oka maps on unramified Riemann domains.

In 1949 S. Hitotsumatsu [9] written in Japanese gave a proof of Oka's Heftungslemma by Weil's integral to solve Levi's Problem in general dimension $n \geq 2$; here he gave no argument of plurisubharmonic exhaustions on pseudoconvex Riemann domains, and so the result might hold for finitely sheeted Riemann domains.

In 1953 Oka [24] proved Theorem 4.16 above by making use of his First and Second Coherence Theorems obtained in [22]: the Third Coherence Theorem was not used there.

In 1954 Bremermann [2] and Norguet [14] independently proved Theorem 4.16 for univalent domains $\Omega \subset \mathbf{C}^n$ with general $n \geq 2$, generalizing Oka's Heftungslemma by means of Weil's integral, similarly to Hitotsumatsu [9].

⁸⁾ They are now available in [25].

Concluding Remark (Problem). It is interesting to learn that Oka invented and proved three fundamental coherence theorems by means of Weierstrass' Preparation Theorem in order to treat the pseudoconvexity problem on singular ramified Riemann domains. Levi's Problem for ramified domains has a counter-example (Fornæss [5]), but in the same time there is a positive case for which Levi's Problem is affirmative ([13]). Therefore, it is an interesting problem to find:

What is necessary and/or sufficient for the validity of Levi's Problem on a ramified Riemann domain X over \mathbf{C}^n ? :

This is open even when X is non-singular.

References

- [1] A. Andreotti and R. Narasimhan, Oka's Heftungslemma and the Levi problem for complex spaces, Trans. Amer. Math. Soc. **111** (1964), 345–366.
- [2] H.J. Bremermann, Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränderlichen, Math. Ann. **128** (1954), 63–91.
- [3] H. Cartan, Idéaux et modules de fonctions analytiques de variables complexes Bull. Soc. Math. France **78** (1950), 29–64.
- [4] J.-P. Demailly, Complex Analytic and Differentiable Geometry, 2012, www-fourier.ujf-grenoble.fr/~demailly/ .
- [5] J.E. Fornæss, A counterexample for the Levi problem for branched Riemann domains over \mathbf{C}^n , Math. Ann. **234** (1978), 275–277.
- [6] H. Grauert, On Levi's problem and the imbedding of real-analytic manifolds, Ann. Math. **68** (1958), 460–472.
- [7] H. Grauert and R. Remmert, Theorie der Steinschen Räume, Grundle. Math. Wiss. 227 Springer-Verlag, Berlin, 1977: Translated to English by A. Huckleberry, Theory of Stein Spaces, Springer-Verlag, Berlin, 1979: Translated to Japanese by K. Miyajima, Stein Kukan Ron, Springer, Tokyo, 2009D
- [8] R.C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall, 1965.
- [9] S. Hitotsumatsu, On Oka's Heftungs Theorem (Japanese), Sugaku **1** (4) (1949), 304–307, Math. Soc. Jpn.
- [10] T. Nishino, Function Theory in Several Complex Variables (in Japanese), The University of Tokyo Press, Tokyo, 1996; Translation into English by N. Levenberg and H. Yamaguchi, Amer. Math. Soc. Providence, R.I., 2001.
- [11] J. Noguchi, Another direct proof of Oka's Theorem (Oka IX), J. Math. Sci. Univ. Tokyo **19** (2012), 1–15.
- [12] J. Noguchi, Tahensu Kaiseiki Kansuron (in Japanese), Asakura-Shoten, Tokyo, 2013: Translated into English, Analytic Function Theory of Several Variables – Elements of Oka's Coherence, Springer, Singapore, 2016.
- [13] J. Noguchi, Inverse of Abelian integrals and ramified Riemann domains, Math. Ann. **367** No. 1 (2017), 229–249.

- [14] F. Norguet, Sur les domaines d'holomorphie des fonctions uniformes de plusieurs variables complexes (Passage du local au global), *Bull. Soc. Math. France* **82** (1954), 137–159.
- [15] K. Oka, Sur les fonctions analytiques de plusieurs variables– I– Domaines convexes par rapport aux fonctions rationnelles, *J. Sci. Hiroshima Univ. Ser. A* **6** (1936), 245–255.
- [16] K. Oka, Sur les fonctions analytiques de plusieurs variables– II – Domaines d'holomorphie, *J. Sci. Hiroshima Univ. Ser. A* **7** (1937), 115–130.
- [17] K. Oka, Sur les fonctions analytiques de plusieurs variables– III – Deuxième problème de Cousin, *J. Sci. Hiroshima Univ.* **9** (1939), 7–19.
- [18] K. Oka, Sur les fonctions analytiques de plusieurs variables– IV – Domaines d'holomorphie et domaines rationnellement convexes, *Jpn. J. Math.* **17** (1941), 517–521.
- [19] K. Oka, Sur les fonctions analytiques de plusieurs variables– V – L'intégrale de Cauchy, *Jpn. J. Math.* **17** (1941), 523–531.
- [20] K. Oka, Sur les domaines pseudoconvexes, *Proc. Imperial Acad. Tokyo* **17** (1941), 7-10.
- [21] K. Oka, Sur les fonctions analytiques de plusieurs variables – VI Domaines pseudoconvexes, *Tôhoku Math. J.* **49** (1942), 15-52.
- [22] K. Oka, Sur les fonctions analytiques de plusieurs variables – VII Sur quelques notions arithmétiques, *Bull. Soc. Math. France* **78** (1950), 1-27
- [23] K. Oka, Sur les fonctions analytiques de plusieurs variables – VIII Lemme fondamental, *J. Math. Soc. Jpn.* **3** (1951) No. 1, 204-214, No. 2, 259-278.
- [24] K. Oka, Sur les fonctions analytiques de plusieurs variables – IX Domaines finis sans point critique intérieur, *Jpn. J. Math.* **23** (1953), 97-155.
- [25] K. Oka, Posthumous Papers of Kiyoshi Oka, Eds. T. Nishino and A. Takeuchi, Kyoto, 1980–1983: Oka Kiyoshi Collection, Library of Nara Women's University, URL "<http://www.lib.nara-wu.ac.jp/oka/>".

Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Meguro-ku, Tokyo 153-8914
Japan
e-mail: noguchi@ms.u-tokyo.ac.jp